Published by Institute of Physics Publishing for SISSA/ISAS
Received: *September 29, 2004*
Accepted: *October 27, 2004*

Gerbes, M5-brane anomalies and E_8 gauge theory

Paolo Aschieri[abc] **and Branislav Jurčo**[bc]

[a] *Dipartimento di Scienze e Tecnologie Avanzate, Universitá del Piemonte Orientale and INFN*
Via Bellini 25/G, 15100 Alessandria, Italy
[b] *Max-Planck-Institut für Physik*
Föhringer Ring 6, D-80805, München, Germany
[c] *Sektion Physik, Universität München*
Theresienstr. 37, D-80333 München, Germany
E-mail: aschieri@theorie.physik.uni-muenchen.de, jurco@theorie.physik.uni-muenchen.de

Abstract: Abelian gerbes and twisted bundles describe the topology of the NS 3-form gauge field strength H. We review how they have been usefully applied to study and resolve global anomalies in open string theory. Abelian 2-gerbes and twisted nonabelian gerbes describe the topology of the 4-form field strength G of M-theory. We show that twisted nonabelian gerbes are relevant in the study and resolution of global anomalies of multiple coinciding M5-branes. Global anomalies for one M5-brane have been studied by Witten and by Diaconescu, Freed and Moore. The structure and the differential geometry of twisted nonabelian gerbes (i.e. modules for 2-gerbes) is defined and studied. The nonabelian 2-form gauge potential living on multiple coinciding M5-branes arises as curving (curvature) of twisted nonabelian gerbes. The nonabelian group is in general $\tilde{\Omega}E_8$, the central extension of the E_8 loop group. The twist is in general necessary to cancel global anomalies due to the nontriviality of the 11-dimensional 4-form field strength G and due to the possible torsion present in the cycles the M5-branes wrap. Our description of M5-branes global anomalies leads to the D4-branes one upon compactification of M-theory to Type IIA theory.

Keywords: Chern-Simons Theories, M-Theory, Anomalies in Field and String Theories, Differential and Algebraic Geometry.

Contents

1. Introduction

The topology of gauge theories with 2-form gauge potentials is a fascinating subject both from the physics and the mathematics perspectives. Consider for example in string theory a stack of n coinciding D-branes, they usually form a U(n) vector bundle, however when there is a topologically nontrivial NS B-field background, this is generally no more the case. Cancellation of global anomalies requires the U(n) bundle to be *twisted* in order to accommodate for the nontrivial topology of the B-field. Thus the study of D-brane charges in the presence of nontrivial backgrounds leads to generalize the usual notion of fibre bundle. A twisted U(n) bundle has transition functions G_{ij} that satisfy the twisted cocycle relations $G_{ij}G_{jk}G_{ki} = \lambda_{ijk}$, where λ_{ijk} are U(1) valued functions. It follows that λ_{ijk} satisfy the cocycle relations $\lambda_{ijk}\lambda_{jkl}^{-1}\lambda_{ikl}\lambda_{ijl}^{-1} = 1$, this is the characteristic property of the transition functions of a bundle gerbe. In short, bundle gerbes, or simply gerbes, are a higher version of line bundles, and the gauge potential for these structures is the 2-form B in the same way as the connection 1-form A is the gauge potential associated with line bundles. As we have sketched, associated with a gerbe we have a twisted bundle (also called gerbe module). The fact that a stack of D-branes in a nontrivial background forms a twisted bundle was studied in [1], confirmed using worldsheet global anomalies in [2] and further generalized using twisted K-theory in [3]–[5].

JHEP10(2004)068

The structure of gauge theories with 3-form gauge potentials is similarly fascinating and rich, the corresponding geometrical structure is that of a 2-gerbe (if the 4-form field-strength is integral). A main motivation for studying these structures is provided by the 3-form C-field of 11-dimensional supergravity. In particular it is interesting to study which M5-brane configurations are compatible with a topologically nontrivial C-field. By requiring the vanishing of global anomalies, topological aspects of the partition function of a single M5-brane have been studied in [6, 7], and in the presence of a nontrivial background in [7] and in [8, 9]. We refer to [10] for the underlying mathematical structures.

In this note we define twisted nonabelian gerbes, these are a higher version of twisted bundles, we study their properties and show that they are associated with abelian 2-gerbes (they are 2-gerbe modules). Using global anomalies cancellation arguments we then see that the geometrical structure underlying a stack of M5-branes is indeed that of a twisted nonabelian gerbe. The associated 2-gerbe is constructed from the C-field data. The twist is necessary in order to accommodate for the nontrivial topology of the C-field. A twisted nonabelian gerbe is (partly) characterized by a nonabelian 2-form gauge potential, in the case of a single M5-brane this becomes the abelian chiral gauge potential of the self-dual 3-form on the M5-brane. Moreover, an M5-brane becomes a D4-brane upon the appropriate compactification of M-theory to Type IIA string theory, and correspondingly the 2-gerbe becomes a gerbe, and the twisted gerbe becomes a twisted bundle. A particular case is when the 2-gerbe is trivial, then the stack of M5-branes gives a nonabelian gerbe. This corresponds in Type IIA to a stack of D-branes forming a bundle. The differential geometry of nonabelian gerbes has been studied at length in [11] (using algebraic geometry) and in [12] (using differential geometry).

It may sound strange to discuss the physical (string theory) relevance of twisted nonabelian gerbes before studying that of the easier case of nonabelian gerbes. This route is however dictated by anomaly cancellation arguments and by the strong analogy between M5-branes in M-theory (with open M2-branes ending on them) and D-branes in Type IIA (with open strings ending on them). Indeed, as we emphasise in section 3, the study of global open string anomalies in the presence of a closed string NS B-field background is enough to conclude that there must be a U(1) gauge potential on a D-brane, and that therefore a D-brane configuration is associated with a line bundle. Even more, if the NS 3-form field H is torsion class (i.e. it is trivial in real De Rham cohomology but not in integer cohomology) then we are obliged to consider coinciding branes forming twisted $\mathrm{U}(n)$ bundles, and this implies that $\mathrm{U}(N)$ bundles also arise for coinciding branes in the previous case where B is torsionless. Similarly, nontrivial backgrounds in M-theory, giving rise to torsion classes, force us to describe the configuration of a stack of M5-branes via twisted nonabelian gerbes. Nonabelian gerbes are then recovered as a special case.

The knowledge of the topology of coinciding M5-branes is a first step toward the formulation of the dynamics of these nonabelian gauge fields. Indeed the full structure of a (twisted) nonabelian gerbe is considerably richer than just a local nonabelian 2-form gauge potential, for example we also have a local 1-form gauge potential and its corresponding

2-form field strength. It is using all these gauge potentials and their gauge transformations (analyzed in section 4) that one can attack the problem of constructing an action describing the dynamics of a stack of M5-branes.

A prominent role in nonabelian gerbes in M-theory is played by the E_8 group. Indeed, for topological considerations, the 2-form gauge potential can be always thought to be valued in ΩE_8, the E_8 loop group, and for twisted nonabelian gerbes in $\tilde{\Omega} E_8$, the central extension of the E_8 loop group. This is so because of the simple homotopy structure of E_8. This corresponds to the fact, exploited in [15], and recalled at the end of this paper, that in Type IIA theory a stack of D-branes gives in general a twisted $\tilde{\Omega} E_8$ bundle, so that at least for topological considerations we can consider the gauge potential to be $\tilde{\Omega} E_8$ valued. This adds to the growing evidence that E_8 plays a main role in M-theory. For example the subtle topology of the 3-form C-field is conveniently described considering it as a composite field, obtained via E_8 valued 1-form gauge potentials, roughly we have $C \sim CS(A_i) = \mathrm{Tr}(A_i dA_i) + \frac{2}{3}\,\mathrm{Tr}(A_i^3)$. Gauge theory with E_8 gauge group has been used in [13], and then, for manifolds with boundary, in [8] in order to globally define the Chern-Simons topological term $\Phi(C) \sim \int \frac{1}{6} C \wedge G \wedge G$. It has been shown in [14] to nicely confirm the K-theory formalism in Type IIA theory upon compactification of M-theory. For further work in this direction see for example [15, 16]. Another instance where E_8 gauge theory appears in M-theory is in Hořava-Witten [17]. Finally it is well known that exceptional groups duality symmetries appear after compactification of supergravity theories, and it has been proposed that these symmetries follow from a hidden E_{11} symmetry of 11-dimensional supergravity [18].

It is interesting to notice that the E_8 formulation of the C-field is not the only one, in particular in [19] another formulation related to $OSp(1|32)$ gauge theory was studied, and is currently investigated, see for example [20]. It might well be that a relation between these two different descriptions can lead to a further understanding of the possibly dynamical role of the E_8 gauge theory.

This paper is organized as follows. Section 2 is a review of abelian gerbes. There are many ways of introducing these structures (see [21] for a recent and nice introduction to the subject), we choose a minimal approach, mainly focusing on Deligne cohomology classes [22], these are a refinement of integral cohomology. Gerbes are then a geometric realizations of Deligne classes. They are equivalent to differential characters, also called Cheeger-Simons characters [23], in this case it is the holonomy of these higher order bundles that is emphasized.

Global worldsheet anomalies of open strings ending on D-branes where studied in [24]; in section 3 we use gerbes in order to construct anomaly free worldsheet actions of strings ending on multiple coinciding D-branes. We mainly follow [2] and [5], but also uncover some details (especially about gauge transformations), simplify the presentation when torsion is present, and emphasize that the gauge fields on the branes can be inferred just from the NS B-field in the bulk.

Section 4 defines and studies twisted nonabelian gerbes. We then give an explicit construction using the loop group of E_8; we also see that any twisted nonabelian gerbe can be realized by lifting an E_8 bundle.

Section 5 uses twisted nonabelian gerbes in order to describe a stack of M5-branes.

JHEP10(2004)068

2. Gerbes

2.1 Abelian 1-Gerbes

Line bundles can be described using transition functions. Consider a cover $\{O_i\}$ of the base space M, then a line bundle is given by a set of U(1) valued smooth transition functions $\{\lambda_{ij}\}$ that satisfy $\lambda_{ij} = \lambda_{ji}^{-1}$ and that on triple overlaps $O_{ijk} = O_i \cap O_j \cap O_k$ satisfy the cocycle condition

$$\lambda_{ij}\lambda_{jk} = \lambda_{ik}\,. \tag{2.1}$$

In the same spirit, a connection on a line bundle is a set of one-forms $\{\alpha_i\}$ on O_i such that on double overlaps $O_{ij} = O_i \cap O_j$,

$$\alpha_i = \alpha_j + \lambda_{ij} d\lambda_{ij}^{-1}\,. \tag{2.2}$$

Actually we are interested only in isomorphic classes of line bundles with connection, indeed all physical observables are obtained from Wilson loops, and these cannot distinguish between a bundle with connection (λ_{ij}, α_i) and an equivalent one $(\lambda'_{ij}, \alpha'_i)$, that by definition satisfies

$$\lambda'_{ij} = \tilde{\lambda}_i \lambda_{ij} \tilde{\lambda}_j^{-1}\,, \qquad \alpha'_i = \alpha_i + \tilde{\lambda}_i d\tilde{\lambda}_i^{-1}\,, \tag{2.3}$$

where $\tilde{\lambda}_i$ are U(1) valued smooth functions on O_i. We are thus led to consider the class $[\lambda_{ij}, \alpha_i]$, of all couples (λ_{ij}, α_i) that satisfy (2.2), and where $(\lambda_{ij}, \alpha_i) \sim (\lambda'_{ij}, \alpha'_i)$ iff (2.3) holds. The space of all these classes (called Deligne classes) is the Deligne cohomology group $H^1(M, \mathcal{D}^1)$. Wilson loops for the Deligne class $[\lambda_{ij}, \alpha_i]$ are given in subsection 2.4.

Similarly we can consider the Deligne class $[\lambda_{ijk}, \alpha_{ij}, \beta_i] \in H^2(M, \mathcal{D}^2)$ where now $\lambda_{ijk} : O_{ijk} \to \mathrm{U}(1)$ is totally antisymmetric in its indices, $\lambda_{ijk} = \lambda_{jik}^{-1} = \lambda_{kij}$ etc., and it satisfies the cocycle condition on triple overlaps

$$\lambda_{ijk}\lambda_{jkl}^{-1}\lambda_{ikl}\lambda_{ijl}^{-1} = 1\,, \tag{2.4}$$

while the connection one-form $\{\alpha_{ij}\}$ satisfies on O_{ijk}

$$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} + \lambda_{ijk} d\lambda_{ijk}^{-1} = 0 \tag{2.5}$$

and the curving two-form $\{\beta_i\}$ satisfies on O_{ij}

$$\beta_i - \beta_j + d\alpha_{ij} = 0\,. \tag{2.6}$$

The triple $(\lambda_{ijk}, \alpha_{ij}, \beta_i)$ gives the zero Deligne class if

$$(\lambda_{ijk}, \alpha_{ij}, \beta_i) = D(\tilde{\lambda}_{ij}, \tilde{\alpha}_i)\,, \tag{2.7}$$

where D is the Deligne coboundary operator, and $\tilde{\lambda}_{ij} : O_{ij} \to \mathrm{U}(1)$ are smooth functions and $\tilde{\alpha}_i$ are smooth one-forms on O_i. Explicitly (2.7) reads[1]

$$\lambda_{ijk} = \tilde{\lambda}_{ik}\tilde{\lambda}_{jk}^{-1}\tilde{\lambda}_{ij}^{-1} \tag{2.8}$$

$$\alpha_{ij} = -\tilde{\alpha}_i + \tilde{\alpha}_j + \tilde{\lambda}_{ij} d\tilde{\lambda}_{ij}^{-1}\,, \tag{2.9}$$

$$\beta_i = d\tilde{\alpha}_i\,. \tag{2.10}$$

[1]The Deligne coboundary operator is $D = \pm\delta + d$, the sign factor in front of the Čech coboundary operator depends on the degree of the form D acts on; it insures $D^2 = 0$.

There is also a geometric structure associated with the triple $(\lambda_{ijk}, \alpha_{ij}, \beta_i)$, it is that of (abelian) gerbe [22] or bundle gerbe [33]. Equivalence classes of gerbes with connection and curving are in 1-1 correspondence with Deligne classes, and with abuse of language we say that $[\mathcal{G}] = [\lambda_{ijk}, \alpha_{ij}, \beta_i]$ is the equivalence class of the gerbe $\mathcal{G} = (\lambda_{ijk}, \alpha_{ij}, \beta_i)$. The holonomy of an abelian gerbe is given in subsection 2.4. As before, gauge invariant (physical) quantities can be obtained from the holonomy (Wilson surface), and this depends only on the equivalence class of the gerbe.

Gerbes are also called 1-gerbes in order to distinguish them from 2-gerbes.

2.2 Abelian 2-Gerbes

Following the previous section, for the purposes of this paper, we understand under an abelian 2-gerbe with curvings on M a quadruple $(\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i)$. Here $\lambda_{ijkl} : O_{ijkl} \equiv O_i \cap O_j \cap O_k \cap O_l \to \mathrm{U}(1)$ is a 2-Čech cocycle

$$\lambda_{ijkl}\lambda_{ijlm}\lambda_{jklm} = \lambda_{iklm}\lambda_{ijkm} \qquad \text{on} \quad O_{ijklm}\,, \tag{2.11}$$

and λ_{ijkl} is totally antisymmetric, $\lambda_{ijkl} = \lambda^{-1}_{jikl}$ etc.. Next $\alpha_{ijk} \in \Omega^1(O_{ijk})$, $\beta_{ij} \in \Omega^2(O_{ij})$ and $\gamma_i \in \Omega^3(O_i)$ are a collection of local one, two, and three-forms totally antisymmetric in their respective indices and subject to the following relations:

$$\alpha_{ijk} + \alpha_{ikl} - \alpha_{ijl} - \alpha_{jkl} = \lambda_{ijkl} d\lambda^{-1}_{ijkl} \qquad \text{on} \quad O_{ijk}\,, \tag{2.12}$$

$$\beta_{ij} + \beta_{jk} - \beta_{ik} = d\alpha_{ijk} \qquad \text{on} \quad O_{ijk}\,, \tag{2.13}$$

$$\gamma_i - \gamma_j = d\beta_{ij} \qquad \text{on} \quad O_{ij}\,. \tag{2.14}$$

The equivalence class of the 2-gerbe with curvings $(\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i)$ is given by the Deligne class $[\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i]$, where the quadruple $(\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i)$ represents the zero Deligne class if it is of the form

$$\lambda_{ijkl} = \tilde{\lambda}^{-1}_{ijl}\tilde{\lambda}^{-1}_{jkl}\tilde{\lambda}_{ijk}\tilde{\lambda}_{ikl}\,, \tag{2.15}$$

$$\alpha_{ijk} = \tilde{\alpha}_{ij} + \tilde{\alpha}_{jk} + \tilde{\alpha}_{ki} + \tilde{\lambda}_{ijk} d\tilde{\lambda}^{-1}_{ijk}\,, \tag{2.16}$$

$$\beta_{ij} = \tilde{\beta}_i - \tilde{\beta}_j + d\tilde{\alpha}_{ij}\,, \tag{2.17}$$

$$\gamma_i = d\tilde{\beta}_i\,. \tag{2.18}$$

The above equations are summarized in the expression

$$(\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i) = D\left(\tilde{\lambda}_{ijk}, \tilde{\alpha}_{ij}, \tilde{\beta}_i\right), \tag{2.19}$$

where D is the Deligne coboundary operator, $\tilde{\lambda}_{ijk}$ are $\mathrm{U}(1)$ valued functions on O_{ijk} and $\tilde{\alpha}_{ij}, \tilde{\beta}_i$ are respectively 1- and 2-forms on O_{ij} and on O_i.

The Deligne class $[\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i] \in H^3(M, \mathcal{D}^3)$ (actually the cocycle $\{\lambda_{ijkl}\}$) defines an integral class $\xi \in H^4(M, \mathbb{Z})$; this is the characteristic class of the 2-gerbe. Moreover $[\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i]$ defines the closed integral 4-form

$$\frac{1}{2\pi i} G = \frac{1}{2\pi i} d\gamma_i\,. \tag{2.20}$$

The 4-form G is a representative of $\xi_{\mathbb{R}}$: the image of the integral class ξ in real de Rham cohomology.

In the same way as abelian 2-gerbes were described above we can define abelian $n-1$-gerbes with curvings using Deligne cohomology classes in $H^n(M,\mathcal{D}^n)$. Correspondingly we have characteristic classes in $H^{n+1}(M,\mathbb{Z})$. The case $n=1$ gives equivalence classes of line bundles with connections, and in this case the characteristic class is the Chern class of the line bundle.

The relation between a Deligne class and its characteristic class leads to the following exact sequence ([22], see [25] for an elementary proof)

$$0 \to \Omega^n_{\mathbb{Z}}(M) \to \Omega^n(M) \to H^n(M,\mathcal{D}^n) \to H^{n+1}(M,\mathbb{Z}) \to 0\,, \tag{2.21}$$

where $\Omega^n_{\mathbb{Z}}(M)$ is the space of closed integral (i.e. whose integration on n-cycles is an integer) n-forms on M. We also have the exact sequence (see for example [26])

$$0 \to H^n(M,\mathrm{U}(1)) \to H^n(M,\mathcal{D}^n) \to \Omega^{n+1}_{\mathbb{Z}}(M) \to 0\,, \tag{2.22}$$

where, as in (2.20), $G \in \Omega^{n+1}_{\mathbb{Z}}(M)$ is the curvature of the $n-1$-gerbe $(\lambda_{i_1,\dots i_{n+1}}, \alpha_{i_1\dots i_n}, \dots,$ $\gamma_i)$.

It is a result of [23], that $H^n(M,\mathcal{D}^n)$ is isomorphic to the space of differential characters $\check{H}^{n+1}(M)$ (Cheeger-Simons characters). An element of $\check{H}^{n+1}(M)$ is a pair (h,F) where h is a homomorphism from the group of n-cycles $\mathbb{Z}_n(M)$ to U(1) and F is an $(n+1)$-form. The pair (h,F) is such that for any $(n+1)$-chain $\mu \in C_{n+1}(M)$ with boundary $\partial\mu$ the following relation holds

$$h(\partial\mu) = \exp\left(\int_\mu F\right). \tag{2.23}$$

The isomorphism with Deligne cohomology groups is given essentially via the holonomy of an $n-1$-gerbe, and $F=G$.

2.3 Special cases

An important example of a 2-gerbe is obtained from an element θ belonging to the torsion subgroup $H^4_{tors}(M,\mathbb{Z})$ of $H^4(M,\mathbb{Z})$. Every torsion element θ is the image of an element $\vartheta \in H^3(M,\mathbb{Q}/\mathbb{Z})$ via the Bockstein homomorphism $\beta : H^3(M,\mathbb{Q}/\mathbb{Z}) \to H^4(M,\mathbb{Z})$ associated with the exact sequence $\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$. As a Čech cocycle ϑ can be represented as a $\mathbb{Q}/\mathbb{Z}$ valued cocycle $\{\vartheta_{ijkl}\}$. Now $\{\vartheta_{ijkl}\}$ can be thought of as a Čech cocycle with values in U(1) valued functions on O_{ijkl}, we have of course $d\vartheta_{ijkl}=0$ and we can thus consider the 2-gerbe $(\vartheta_{ijkl},0,0,0)$. The equivalence class of this 2-gerbe is the Deligne class

$$[\vartheta_{ijkl},0,0,0]\,; \tag{2.24}$$

it depends only on $\theta=\beta(\vartheta)$, the characteristic class of this Deligne class.

Given a globally defined 3-form $C \in \Omega^3(M)$ we can construct the Deligne class

$$[1,0,0,C|_{O_i}]\,. \tag{2.25}$$

Accordingly with (2.21) it has trivial characteristic class and it is the zero Deligne class iff $C \in \Omega^3_{\mathbb{Z}}(M)$. Indeed in this case we can write $(1,0,0,C|_{O_i}) = D(\lambda_{ijk},\alpha_{ij},\beta_i)$ where $(\lambda_{ijk},\alpha_{ij},\beta_i)$ is a 1-gerbe with curvature C. Following [5], Deligne classes like $[1,0,0,C|_{O_i}]$

will be called *trivial*. Notice that a trivial characteristic class is the same as a zero characteristic class while a trivial Deligne class is usually not a zero Deligne class.

These two constructions obviously also apply to n-gerbes. In particular we have the torsion 1-gerbe class

$$[\vartheta_{ijk}, 0, 0]\,, \tag{2.26}$$

associated with the element $\theta \in H^3_{tors}(M, \mathbb{Z})$. Similarly we have the trivial 1-gerbe class $[1, 0, B|_{O_i}]$ associated with a globally defined 2-form $B \in \Omega^2(M)$.

Another family of 2-gerbes, the so-called Chern-Simons 2-gerbes, comes from a principal G-bundle $P_G \to M$. Its characteristic class is the first Pontryagin class of P_G, $p_1 \in H^4(M, \mathbb{Z})$. If G is connected, simply connected and simple and if A is a connection on P_G, given locally by a collection of Lie(G)-valued one-forms A_i, then the image of p_1 in real cohomology equals the cohomology class of $\mathrm{Tr}\, F^2$, and we can identify the local three-forms γ_i with the Chern-Simons forms $CS(A_i)$,

$$\gamma_i = \mathrm{Tr}(A_i dA_i) + \frac{2}{3}\,\mathrm{Tr}(A_i^3)\,.$$

The two-forms β_{ij} and α_{ijkl} and the Čech cocycle λ_{ijkl} can in principle be obtained by solving descent equations [27] (see also [28]). We will denote the 2-gerbe obtained this way as $CS(p_1)$.

Notice that if $\mathrm{dim} M \leq 15$, then there is a one to one correspondence between $H^4(M, \mathbb{Z})$ and isomorphism classes of principal E_8 bundles on M, see [29] for an elementary proof. This follows from the fact that the first nontrivial homotopy group of E_8, except $\pi_3(E_8) = \mathbb{Z}$, is $\pi_{15}(E_8)$. We then have that up to the 14th-skeleton E_8 is homotopy equivalent to the Eilenberg-MacLane space $K(\mathbb{Z}, 3)$ (defined as the space whose only nontrivial homotopy group is $\pi_3(K(\mathbb{Z}, 3)) = \mathbb{Z}$). Similarly up to the 15th-skeleton we have $BE_8 \sim K(\mathbb{Z}, 4)$, where BE_8 is the classifying space of E_8 principal bundles. For the homotopy classes of maps from M to E_8 it then follows that $[M, E_8] = [M, K(\mathbb{Z}, 3)] = H^3(M, \mathbb{Z})$ if $\mathrm{dim} M \leq 14$, and similarly $\{$Equivalence classes of E_8 bundles on $M\} = [M, BE_8] = [M, K(\mathbb{Z}, 4)] = H^4(M, \mathbb{Z})$ if $\mathrm{dim} M \leq 15$. Therefore, corresponding to an element $a \in H^4(M, \mathbb{Z})$ we have an E_8 principal bundle $P(a) \to M$ with $p_1(P(a)) = a$ and picking a connection A on $P(a)$ we have a Deligne class, the Chern-Simons 2-gerbe $CS(a)$, with a being its characteristic class.

As in this paper we are mainly concerned with 2-gerbes associated with 5-branes embedded in 11-dimesional spacetime it is worth to recall also the homotopy groups of the groups G_2, Spin_n, F_4, E_6 and E_7. Except π_3 which is of course $\mathbb{Z}$ in each case, the first nonzero ones are $\pi_6(G_2)$, $\pi_7(\mathrm{Spin}_n)$ where $n \geq 7$, $\pi_8(F_4)$, $\pi_8(E_6)$ and $\pi_{11}(E_7)$. So in the case of a 5-brane, with 6-dimensional worldsheet M, we can replace E_8 bundles with G_2, Spin_n where $n \geq 7$, F_4, E_6 or E_7 bundles in the above discussion.

2.4 Holonomy of line bundles, 1-gerbes and 2-gerbes

The holonomy can be associated with any Deligne class. It gives the corresponding differential character for cycles that arise as images of triangulated manifolds. Here we just collect formulas in the case of 0-, 1- and 2-gerbes [30], see also [5].

Line bundles. The holonomy of $[\lambda_{ij}, \alpha_i]$ around a loop $\varsigma : S \to M$ can be calculated splitting S in sufficiently small arches b and corresponding vertices v, such that each $\varsigma(b)$ is completely contained in an open O_i of the cover $\{O_i\}$ of M. The index i depends on the arch e, we thus call it $\rho(e)$, and write $\varsigma(e) \subset O_{\rho(e)}$; we also associate an index $\rho(v)$ with every vertex v and write $\varsigma(v) \subset O_{\rho(v)}$. We then have

$$\mathrm{hol}(\varsigma) = \prod_e \exp \int_e \varsigma^* \alpha_{\rho(e)} \prod_{v \subset e} \lambda^{\sigma_{e,v}}_{\rho(e)\rho(v)}(\varsigma(v)) \,, \tag{2.27}$$

where $\sigma_{e,v} = 1$ if v is the final point of the oriented arch e, and -1 if it is the initial point. Note that the holonomy depends only on the class $[\lambda_{ij}, \alpha_i]$ and not on the representative (λ_{ij}, α_i) or on the splitting of S or the choice of the index map ρ. Of course if the loop is the boundary of a disk, i.e., if $\zeta : D \to M$ is such that $\zeta|_{\partial D} = \varsigma$, then $\mathrm{hol}(\varsigma) = e^{\int_D \zeta^* F}$.

1-gerbes. We now consider the map $\zeta : \Sigma \to M$ where Σ is a 2-cycle that we triangulate with faces, edges and vertices, denoted f, e and v. The faces f inherit the orientation of Σ, we also choose an orientation for the edges e. It is always possible to choose a triangulation subordinate to the open cover O_i of M and define an index map ρ which maps faces, edges and vertices, into the index set of the covering of M in a way that $\zeta(f) \in O_{\rho(f)}$, etc.. The holonomy of the class $[\lambda_{ijk}, \alpha_{ij}, \beta_i]$ is then

$$\mathrm{hol}(\zeta) = \prod_f \exp \int_f \zeta^* \beta_{\rho(f)} \prod_{e \subset f} \exp \int_e \zeta^* \alpha_{\rho(f)\rho(e)} \prod_{v \subset e \subset f} \lambda_{\rho(f)\rho(e)\rho(v)}(\zeta(v)) \,, \tag{2.28}$$

where it is understood that $\alpha_{\rho(f)\rho(e)}$ appears with the opposite sign if f and e have incompatible orientations. Similarly the inverse of $\lambda_{\rho(f)\rho(e)\rho(v)}$ appears if f and e have incompatible orientations or if v is not the final vertex of e. As before the holonomy depends only on the equivalence class of the gerbe and not on the chosen representative gerbe. It is also independent from the choice of triangulation, of index map ρ and of orientation of the edges.

2-gerbes. We now consider the map $\xi : \Gamma \to M$ where Γ is a 3-cycle. We triangulate it with tetrahedrons, faces, edges and vertices, denoted t, f, e and v. The triangulation is chosen to be subordinate to the open cover $\{O_i\}$ of M. The index map ρ now maps tetrahedrons, faces etc. into the index set of the covering $\{O_i\}$. The formula for the holonomy of the class $[\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i]$ is

$$\mathrm{hol}(\xi) = \prod_t \exp \int_t \xi^* \gamma_{\rho(t)} \prod_{f \subset t} \exp \int_f \xi^* \beta_{\rho(t)\rho(f)} \times \tag{2.29}$$

$$\times \prod_{e \subset f \subset t} \exp \int_e \xi^* \alpha_{\rho(t)\rho(f)\rho(e)} \prod_{v \subset e \subset f \subset t} \lambda_{\rho(t)\rho(f)\rho(e)\rho(v)}(\xi(v)) \,. \tag{2.30}$$

3. Open strings worldsheet anomalies, 1-gerbes and twisted bundles

It is commonly said that the low energy effective action of a stack of n branes is a U(n) Yang-Mills theory. Therefore n coinciding branes are associated with a U(n) bundle. More in general, in the presence of a nontrivial H field we do not have a U(n) bundle, rather a twisted one, i.e. we have a PU(n) bundle that cannot be lifted to a U(n) one, i.e. the PU(n) transition functions g_{ij} cannot be lifted to U(n) transition functions G_{ij} such that under the projection U$(n) \to$ PU(n) we have $G_{ij} \to g_{ij}$ and such that the cocycle condition $G_{ij}G_{jk}G_{ki} = 1$ holds. The twisting is necessary in order to cancel global worldsheet anomalies for open strings ending on D-branes. In this section we study this mechanism. Consider for simplicity the path integral for open bosonic string theory in the presence of D-branes wrapping a cycle Q inside spacetime M and with a given closed string background metric g and NS three form H. We have

$$\int \mathcal{D}\zeta \; e^{i\int_\Sigma L_{\rm NG}}\, e^{\int_\Sigma \zeta^* d^{-1}H}\, \mathrm{Tr}\, \mathrm{hol}^{-1}_{\partial\Sigma}(\zeta^* A) \tag{3.1}$$

here $\zeta : \Sigma \to M$ are maps from the open string worldsheet Σ to the target spacetime M such that the image of the boundary $\partial\Sigma$ lives on Q, we denote by $\Sigma_Q(M)$ this space, $L_{\rm NG}$ is the Nambu-Goto lagrangian, $\int_\Sigma \zeta^* d^{-1}H$ is locally given by $\int_\Sigma \zeta^* B = \int_\Sigma \varepsilon^{\alpha\beta} B_{MN}\partial_\alpha X^M \partial_\beta X^N$ and is the topological coupling of the open string to the NS field, and Tr hol$_\gamma(\zeta^* A)$ is the trace of the holonomy (Wilson loop) around the boundary $\partial\Sigma$ of the nonabelian gauge field A that lives on the n coincident D-branes wrapping Q. Now, while the exponential of the Nambu-Goto action is a well defined function from $\Sigma_Q(M)$ to the circle U(1), the other U(1) factor $e^{\int_\Sigma \zeta^* B}$ is more problematic because only $H = dB$ is globally defined, while $B =$ "$d^{-1}H$" is defined only locally. In order to define this term we need to know not only the integral cohomology of H but the full Deligne class $[\mathcal{G}] = [\lambda_{ijk}, \alpha_{ij}, \beta_i]$ whose curvature is H. We call the gerbe $\mathcal{G}|_Q$ trivial if its class $[\mathcal{G}|_Q]$ is trivial i.e. if (cf. (2.25)): 1) H restricted to Q is cohomologically trivial, that is it exists a B_Q globally defined on Q such that

$$H|_Q = dB_Q\,, \tag{3.2}$$

and 2) the characteristic class ξ of the gerbe is trivial ($H|_Q$ is trivial also in integer cohomology). It turns out that if $\mathcal{G}|_Q$ is trivial, then defining

$$e^{\int_\Sigma \zeta^* d^{-1}H} \equiv \mathrm{hol}(\Sigma\sharp D)\, e^{\int_D \tilde\zeta^* B_Q} \tag{3.3}$$

we have a well defined function on $\Sigma_Q(M)$. Here D is the disk and $\tilde\zeta : D \to Q$ is such that the boundary of $\tilde\zeta(D)$ coincides with the boundary of $\zeta(\Sigma)$ (we have assumed Q simply connected and $\partial\Sigma$ a single loop). Moreover hol$(\Sigma\sharp D) \equiv$ hol$(\zeta\sharp\tilde\zeta)$ is the holonomy of the closed surface $\zeta(\Sigma)\sharp\tilde\zeta(D)$ obtained by gluing together $\zeta(\Sigma)$ and $\tilde\zeta(D)$ (and thus in particular it is obtained by changing the orientation of D).

The two terms hol$(\Sigma\sharp D)$ and $e^{\int_D \tilde\zeta^* B}$ depend on $\tilde\zeta : D \to Q$ and are not functions on $\Sigma_Q(M)$ but respectively sections of a U(1) (or line) bundle $\partial^{-1}\mathcal{L}^{-1}_{[\mathcal{G}|_Q]}$ on $\Sigma_Q(M)$ and of the opposite bundle $\partial^{-1}\mathcal{L}_{[\mathcal{G}|_Q]}$ on $\Sigma_Q(M)$ so that indeed their product is a well defined function on $\Sigma_Q(M)$.[2] The bundle $\partial^{-1}\mathcal{L}_{[\mathcal{G}|_Q]} \to \Sigma_Q(M)$ is constructed from the 1-gerbe

[2] A section of a canonically trivial bundle such as $\mathcal{L}^{-1}\mathcal{L} \to \Sigma_Q(M)$ is automatically a global function on $\Sigma_Q(M)$ because $\mathcal{L}^{-1}\mathcal{L} \to \Sigma_Q(M)$ has the canonical section 1 (locally 1 is the product of an arbitrary section s^{-1} of $\mathcal{L}^{-1}$ and of the corresponding section s of $\mathcal{L}$) and two global sections define a U(1) function on the base space

class. Without entering this construction (described after eq. (3.7)) we can directly see that expression (3.3) is a well defined function on $\Sigma_Q(M)$ by showing its independence from the choice of the map $\tilde{\zeta}$. Given another map $\tilde{\zeta}'$ we have

$$\mathrm{hol}(\zeta\sharp\tilde{\zeta})/\mathrm{hol}(\zeta\sharp\tilde{\zeta}') = \mathrm{hol}(\tilde{\zeta}'\sharp\tilde{\zeta}) = e^{\int_D \tilde{\zeta}'^* B_Q - \int_D \tilde{\zeta}^* B_Q}\,, \tag{3.4}$$

where the first equality is the holonomy gluing property and the last equality holds because the integral of B_Q on $\tilde{\zeta}'\sharp\tilde{\zeta}$ equals the holonomy of the gerbe since B_Q gives a gerbe $(1, 0, B_Q)$ on Q equivalent to $\mathcal{G}|_Q$: $[1, 0, B_Q] = [\mathcal{G}|_Q]$.

Expression (3.3) depends on the equivalence class of the initial gerbe $\mathcal{G}$ and also on B_Q, not just on $[1, 0, B_Q]$. Had we chosen a different 2-form B'_Q such that $[1, 0, B'_Q] = [\mathcal{G}|_Q] = [1, 0, B_Q]$, then the result would have differed by the phase

$$e^{\int_D \tilde{\zeta}^*(B'_Q - B_Q)}\,, \tag{3.5}$$

where $\frac{1}{2\pi i}\omega \equiv \frac{1}{2\pi i}(B'_Q - B_Q)$ is a closed integral 2-form, recall (2.22). In order to absorb this extra phase (this gauge transformation) we have to consider the last term in (3.1): $\mathrm{Tr}\ \mathrm{hol}_{\partial\Sigma}(\zeta^* A)$. This is a well defined U(1)-valued function on $\Sigma_Q(M)$ and A is a true $\mathrm{U}(n)$ connection on a nonabelian bundle on Q, with Tr the trace in the fundamental of $\mathrm{U}(n)$. Under the gauge transformation $B_Q \to B'_Q = B_Q + \omega$ we have to transform accordingly the $\mathrm{U}(n)$ bundle in order to compensate for the phase factor (3.5). This is obtained considering the new $\mathrm{U}(n)$ bundle with curvature $F' = F + \omega$ obtained by tensoring the initial $\mathrm{U}(N)$ bundle on Q with the U(1) bundle on Q defined by the closed 2-form ω (the definition of this U(1) bundle is unique since we have considered Q simply connected). If we consider just one D-brane we recover the gauge invariance of the total U(1) field $B_Q - F$; the gauge transformations locally read $B_Q \to B_Q + d\Lambda$ and $A \to A + \Lambda$.

In conclusion using anomaly cancellation we have seen that if the open strings couple to the B field, then their ends must couple to a U(1) gauge field A. So far there is no requirement for nonabelian gauge fields.

The situation is more involved if $\mathcal{G}|_Q$ has torsion, i.e. if the three form H restricted to Q is cohomologically trivial, but the characteristic class of the gerbe is nontrivial. In this case (3.3) is not well defined because $[1, 0, B_Q] \neq [\mathcal{G}|_Q]$. However any torsion gerbe can be obtained form a lifting gerbe, i.e. from a gerbe that describes the obstruction of lifting a $\mathrm{PU}(n)$ bundle to a $\mathrm{U}(n)$ one (with appropriate n). We now describe this lifting gerbe and the associated twisted $\mathrm{U}(n)$ bundle. Let $P \to M$ be a $\mathrm{PU}(n)$ bundle and consider the exact sequence $\mathrm{U}(1) \to \mathrm{U}(n) \xrightarrow{\pi} \mathrm{PU}(n)$. Consider an open cover $\{U_\alpha\}$ of $\mathrm{PU}(n)$ with sections $s^\alpha : U_\alpha \subset \mathrm{PU}(n) \to \mathrm{U}(n)$. Consider also a good cover $\{O_i\}$ of M such that each transition function g_{ij} of $P \to M$ has image contained in a U_i (this is always doable, we also fix a map from the couples of indices (i, j) to the α indices). Let $G_{ij} = s^\alpha(g_{ij})$, these are $\mathrm{U}(n)$ valued functions and satisfy:

$$G_{ik} G_{jk}^{-1} G_{ij}^{-1} = \lambda_{ijk}\,,$$

where λ_{ijk} is U(1) valued as is easily seen applying the projection π and using the cocycle relation for the g_{ij} transition functions. We say that G_{ij} are the transition functions for a $\mathrm{U}(n)$ twisted bundle and that the lifting gerbe is defined by the twist λ_{ijk}. It is indeed easy to check that the λ_{ijk} satisfy the cocycle condition on quadruple overlaps O_{ijkl}. A connection for a twisted bundle is a set of 1-forms A_i such that $\alpha_{ij} \equiv -A_i + G_{ij}A_jG_{ij}^{-1} + G_{ij}dG_{ij}^{-1}$ is a connection for the corresponding gerbe (in particular $\pi_* A$ is a connection on the initial $\mathrm{PU}(n)$ bundle P). We restate this construction this way: consider the couple (G_{ij}, A_i), and define

$$\begin{aligned}\mathbf{D}(G_{ij}, A_i) &\equiv \left((\delta G)_{ijk}, (\delta A)_{ij} + G_{ij}dG_{ij}^{-1}, \frac{1}{n}\operatorname{Tr} dA_i\right)\\ &= \left(G_{ik}G_{jk}^{-1}G_{ij}^{-1}, -A_i + G_{ij}A_jG_{ij}^{-1} + G_{ij}dG_{ij}^{-1}, \frac{1}{n}\operatorname{Tr} dA_i\right). \qquad (3.6)\end{aligned}$$

If this triple has abelian entries then it defines a gerbe, and (G_{ij}, A_i) is called a twisted bundle. We also say that the twisted bundle (G_{ij}, A_i) is twisted by the gerbe $\mathbf{D}(G_{ij}, A_i)$. Notice that the nonabelian $\mathbf{D}$ operation becomes the abelian Deligne coboundary operator D if $n = 1$ in $\mathrm{U}(n)$ (cf. (2.7)).

More in general, if $\mathcal{G}|_Q = (\lambda_{ijk}, \alpha_{ij}, \beta_i)|_Q$, is torsion then it follows from the results in [31] that one can always find a twisted bundle (G_{ij}, A_i) such that

$$(\lambda_{ijk}, \alpha_{ij}, \beta_i)|_Q = \mathbf{D}(G_{ij}, A_i) + (1, 0, B_Q)\,, \qquad (3.7)$$

where B_Q is a globally defined abelian 2-form.

We can now correctly define the path integral (3.1). We proceed is three steps.

i) Using the holonomy gluing property it is easy to see that $\mathrm{hol}(\Sigma\sharp D) \equiv \mathrm{hol}(\zeta\sharp\tilde{\zeta})$ is a section of the line bundle $\partial^{-1}\mathcal{L}_{[-\mathcal{G}|_Q]} \to \Sigma_Q(M)$ at the point $\zeta \in \Sigma_Q(M)$. The line bundle $\partial^{-1}\mathcal{L}_{[-\mathcal{G}|_Q]} \to \Sigma_Q(M)$ is the pull back to $\Sigma_Q(M)$ of the line bundle on loop space $\mathcal{L}_{[-\mathcal{G}|_Q]} \to L(Q)$. We characterize $\mathcal{L}_{[-\mathcal{G}|_Q]} \to L(Q)$ (here $-\mathcal{G}|_Q$ is a generic gerbe over Q) by realizing its sections $L(Q) \to \mathcal{L}_{[-\mathcal{G}|_Q]}$ through functions $s : D(Q) \to \mathbb{C}$ where $D(Q)$ is the space of maps from the disk D into Q; the boundaries of these maps are loops in $L(Q)$. The function s is a section of $\mathcal{L}_{[-\mathcal{G}|_Q]} \to L(Q)$ if $s(\tilde{\zeta}) = \mathrm{hol}(\tilde{\zeta}\sharp\tilde{\zeta}')s(\tilde{\zeta}')$ for all $\tilde{\zeta}, \tilde{\zeta}' \in D(Q)$ that are equal on the boundary: $\tilde{\zeta}|_{\partial D} = \tilde{\zeta}'|_{\partial D}$. Expression $\mathrm{hol}(\tilde{\zeta}\sharp\tilde{\zeta}')$ above is the holonomy of $[-\mathcal{G}|_Q]$ on the closed surface $\tilde{\zeta}(D)\sharp\tilde{\zeta}'(D)$ obtained by gluing together $\tilde{\zeta}(D)$ and $\tilde{\zeta}'(D)$.

ii) If we define $\mathcal{T} = \mathbf{D}(G_{ij}, A_i)$, then (3.7) reads $\mathcal{G}|_Q - \mathcal{T} = (1, 0, B_Q)$ and we see that $e^{\int_D \tilde{\zeta}^* B_Q}$ is a section of the line bundle $\mathcal{L}_{[\mathcal{G}|_Q - \mathcal{T}]} \to L(Q)$. From i) and ii) we see that we need a section of the line bundle $\mathcal{L}_{[\mathcal{T}]} \to L(Q)$.

iii) A section of the line bundle $\mathcal{L}_{[\mathcal{T}]} \to L(Q)$ is given by the inverse of the trace of the holonomy of the twisted $\mathrm{U}(n)$ bundle (G_{ij}, A_i). The definition is as follows. The pull back of $\mathcal{T}$ on the disk D via $\tilde{\zeta} : D \to Q$ is trivial since D is two dimensional. We can thus write

$$\mathbf{D}(\tilde{\zeta}^* G_{ij}, \tilde{\zeta}^* A_i) = D(\tilde{\lambda}_{ij}, \tilde{\alpha}_i) + (1, 0, b) \qquad (3.8)$$

so that $(\tilde{\zeta}^* G_{ij}\,\lambda_{ij}^{-1}, \tilde{\zeta}^* A_i - \tilde{\alpha}_i)$ is a true $\mathrm{U}(n)$ bundle. We then define

$$\operatorname{Tr}\mathrm{hol}_{\partial\Sigma}(\zeta^* A) \equiv \operatorname{Tr}\mathrm{hol}_{\partial\Sigma}(\tilde{\zeta}^* A - \tilde{\alpha})e^{-\int_D b}\,, \qquad (3.9)$$

where $\mathrm{Tr\,hol}_{\partial\Sigma}(\tilde{\zeta}^*A - \tilde{\alpha})$ is the trace of the holonomy of the $\mathrm{U}(N)$ bundle $(\tilde{\zeta}^*G_{ij}\,\tilde{\lambda}_{ij}^{-1}, \tilde{\zeta}^*A_i - \tilde{\alpha}_i)$. Note that if we consider the couple (G_{ij}, A_i) in the adjoint representation, then it defines a true $\mathrm{SU}(n)$ bundle. Consistently, if in (3.9) we consider the trace in the adjoint representation instead of the trace in the fundamental, we then obtain the holonomy of this $\mathrm{SU}(n)$ bundle. It is easy to check that definition (3.9) is independent from the choice of the trivialization $\tilde{\lambda}, \tilde{\alpha}$, b and of the map $\tilde{\zeta} : D \to Q$.

We conclude that expression (3.1) is well defined because we have the product of the three sections

$$\mathrm{hol}(\Sigma \sharp D)\; e^{\int_D \tilde{\zeta}^* B_Q}\; \mathrm{Tr\,hol}_{\partial\Sigma}^{-1}(\tilde{\zeta}^*A)\,, \tag{3.10}$$

respectively sections of the bundles $\partial^{-1}\mathcal{L}_{[-\mathcal{G}|_Q]}$, $\partial^{-1}\mathcal{L}_{[\mathcal{G}|_Q - \mathcal{T}]}$ and $\partial^{-1}\mathcal{L}_{[\mathcal{T}]}$ on the base space $\Sigma_Q(M)$ obtained by pulling back the corresponding bundles on the loop space $L(Q)$ via the map $\Sigma_Q(M) \stackrel{\partial}{\to} L(Q)$. The product of these three bundles is canonically trivial. Expression (3.10) depends on the Deligne classes $[\mathcal{G}]$, $[\mathcal{T}]$ and on the potential B_Q; it is easily seen that it does not depend on the choice of the map $\tilde{\zeta} : D \to Q$. In order to obtain a gauge invariant action the gauge transformation $B_Q \to B'_Q = B_Q + \omega$ comes always together with the transformation of the twisted $\mathrm{U}(N)$ bundle (G_{ik}, A_i) obtained by tensoring (G_{ik}, A_i) with the $\mathrm{U}(1)$ bundle on Q defined by the closed 2-form ω.

In conclusion using anomaly cancellation we have seen that if the open strings couple to the B field — more precisely to the gerbe class $[\mathcal{G}]$ — then their ends must couple to a twisted $\mathrm{U}(n)$ gauge field A if on the boundary $\mathcal{G}$ is torsion.

For sake of simplicity, up to now we have omitted spinor fields. In superstring theory, due to the determinant of the spinor fields, we have an extra term entering the functional integral: Pfaff. This is a section of the bundle $\partial(\mathcal{L}_{[\omega_{ijk},0,0]}) \to \Sigma_Q(M)$ where $[\omega_{ijk}, 0, 0]$ is the Deligne class associated with the second Stiefel-Whitney class $\omega_2 \in H^2(Q, \mathbb{Z}_2)$ of the normal bundle of Q [or, which is the same, with its image W_3 in $H^3_{tors}(Q, \mathbb{Z})$]. In this case we consider a $\mathrm{PU}(n)$ bundle $P \to Q$ with curvature two form such that instead of (3.7) the following equation holds, $(\lambda_{ijk}, \alpha_{ij}, \beta_i)|_Q - (\omega_{ijk}, 0, 0) = \mathbf{D}(G_{ij}, A_i) + (1, 0, B_Q)$. Correspondingly, the product

$$\mathrm{Pfaff}\;\; \mathrm{hol}(\Sigma \sharp D)\; e^{\int_D \tilde{\zeta}^* B_Q}\; \mathrm{Tr\,hol}_{\partial\Sigma}^{-1}(\tilde{\zeta}^*A)\,, \tag{3.11}$$

is a well defined function on $\Sigma_Q(M)$ because $\partial^{-1}\mathcal{L}_{[\omega_{ijk},0,0]}\,\partial^{-1}\mathcal{L}_{[-\mathcal{G}|_Q]}\partial^{-1}\mathcal{L}_{[1,0,B|_Q]}\,\partial^{-1}\mathcal{L}_{[\mathcal{T}]}$ is the trivial bundle. We thus arrive at the general condition for a stack of D-branes to be wrapping a cycle Q in M. It is

$$[\lambda_{ijk}, \alpha_{ij}, \beta_i]|_Q - [\omega_{ijk}, 0, 0] = [\mathbf{D}(G_{ij}, A_i)] + [1, 0, B_Q]\,, \tag{3.12}$$

i.e. the stack of D-branes must form a twisted bundle, the twist being given by a gerbe that up to a trivial gerbe is equal to the initial gerbe associated with the 3-form H minus the gerbe obtained from the second Stiefel-Whitney class of the normal bundle of Q. In particular, for the characteristic classes of these gerbes we have,

$$[H]|_Q - W_3 = \xi_{[\mathbf{D}(G_{ij}, A_i)]}\,, \tag{3.13}$$

where $[H]|_Q \equiv \xi_{\mathcal{G}|_Q}$ is the characteristic class of the restriction to Q of the gerbe $\mathcal{G} = (\lambda_{ijk}, \alpha_{ij}, \beta_i)$ associated with the 3-form H, and $W_3 = \beta(\omega_2)$ is the obstruction for having Spin^c structure on the normal bundle of Q, in fact β is the Bockstein homomorphism associated with the short exact sequence $\mathbb{Z} \stackrel{\times 2}{\to} \mathbb{Z} \to \mathbb{Z}_2$.

4. Twisted nonabelian gerbes (2-gerbe modules)

In this section we slightly generalize the notion of twisted bundle (1-gerbe module) and then consider the one degree higher case. In (3.6) twisted U(n) bundles where defined. More generally, consider the central extension:

$$\mathrm{U}(1) \to G \stackrel{\pi}{\to} G/\,\mathrm{U}(1) \tag{4.1}$$

i.e. where U(1) is mapped into the center $Z(G)$ of G. (In the following we will not distinguish between U(1) and its image $\ker\pi \subset G$). A twisted G bundle with connection A and curvature F is a triple (G_{ij}, A_i, F_i) such that

$$F_i = G_{ij} F_j G_{ij}^{-1}\,, \tag{4.2}$$

and such that

$$\begin{aligned}\mathbf{D}_F(G_{ij}, A_i) &\equiv \Big((\delta G)_{ijk}, (\delta A)_{ij} + G_{ij} dG_{ij}^{-1}, dA_i + A_i \wedge A_i - F_i\Big)\\ &= (G_{ik} G_{jk}^{-1} G_{ij}^{-1}, -A_i + G_{ij} A_j G_{ij}^{-1} + G_{ij} dG_{ij}^{-1}, dA_i + A_i \wedge A_i - F_i)\end{aligned} \tag{4.3}$$

has U(1)- and Lie(U(1))-valued entries.

It is not difficult to check that the triple (4.3) defines a gerbe (hint: since the group extension is central, $d((\delta A)_{ij} + G_{ij} dG_{ij}^{-1}) = -dA_i - A_i \wedge A_i + G_{ij}(dA_j + A_j \wedge A_j)G_{ij})$. In the U$(n)$ case there was no need to introduce the extra data of the curvature F because at the Lie algebra level Lie(U(n)) $= Lie(\mathrm{U}(n)/\,\mathrm{U}(1)) \otimes$ Lie(U(1)), so that F was canonically constructed from A.

The notion of twisted 1-gerbe (2-gerbe module) can be introduced performing a similar construction. While twisted nonabelian bundles are described by nonabelian transition functions $\{G_{ij}\}$, twisted nonabelian gerbes are described by transition functions $\{f_{ijk}, \varphi_{ij}\}$ that are respectively valued in G and in $Aut(G)$, $f_{ijk} : O_{ijk} \to G$, $\varphi_{ij} : O_{ij} \to Aut(G)$, and where the action of φ_{ij} on U(1) is trivial: $\varphi_{ij}|_{\mathrm{U}(1)} = id$. The twisted cocycle relations now read

$$\lambda_{ijkl} = f_{ikl}^{-1} f_{ijk}^{-1} \varphi_{ij}(f_{jkl}) f_{ijl}\,, \tag{4.4}$$

$$\varphi_{ij}\varphi_{jk} = Ad_{f_{ijk}} \varphi_{ik}\ , \tag{4.5}$$

where $\{\lambda_{ijkl}\}$ is U(1)-valued. It is not difficult to check that $\{\lambda_{ijkl}\}$ is a Čech 3-cocycle and thus defines a 2-gerbe (without curvings). This cocycle may not satisfy the antisymmetry property in its indices, this however can always be achieved by a gauge transformation with a trivial cocycle. In the particular case $\lambda_{ijkl} = 1$ equations (4.4), (4.5) define a nonabelian 1-gerbe (without curvings).

One can also consider twisted gerbes with connections 1-forms: $(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i)$ where $a_{ij} \in \mathrm{Lie}(G) \otimes \Omega^1(O_{ij})$, $\mathcal{A}_i \in \mathrm{Lie}(Aut(G)) \otimes \Omega^1(O_i)$, and twisted gerbes with curvings:

$$(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i, B_i, d_{ij}, H_i)\,, \tag{4.6}$$

where B_i, d_{ij} are 2-forms and H_i 3-forms, all of them Lie(G)-valued; $B_i \in \mathrm{Lie}(G) \otimes \Omega^2(O_i)$, $d_{ij} \in \mathrm{Lie}(G) \otimes \Omega^2(O_{ij})$, $H_i \in \mathrm{Lie}(G) \otimes \Omega^3(O_i)$. Before defining a twisted 1-gerbe we need to introduce some more notation. Given an element $X \in \mathrm{Lie}(Aut(G))$, we can construct a map (a 1-cocycle) $T_X : G \to \mathrm{Lie}(G)$ in the following way,

$$T_X(h) \equiv [he^{tX}(h^{-1})]\,,$$

where $[he^{tX}(h^{-1})]$ is the tangent vector to the curve $he^{tX}(h^{-1})$ at the point 1_G; if X is inner, i.e. $X = ad_Y$ with $Y \in \mathrm{Lie}(G)$, then $e^{tX}(h^{-1}) = e^{tY}h^{-1}e^{-tY}$ and we simply have $T_X(h) = T_{ad_Y}(h) = hYh^{-1} - Y$. Given a Lie$(Aut(G))$-valued form $\mathcal{A}$, we write $\mathcal{A} = \mathcal{A}^\rho X^\rho$ where $\{X^\rho\}$ is a basis of Lie$(Aut(G))$. We then define $T_{\mathcal{A}}$ as

$$T_{\mathcal{A}} \equiv \mathcal{A}^\rho T_{X^\rho}\,. \tag{4.7}$$

We use the same notation $T_{\mathcal{A}}$ for the induced map on Lie(G). Now we extend this map to allow $T_{\mathcal{A}}$ to act on a Lie(G)-valued form $\eta = \eta^\alpha Y^\alpha$, where $\{Y^\alpha\}$ is a basis of Lie(G), by $T_{\mathcal{A}}(\eta) = \eta^\alpha \wedge T_{\mathcal{A}}(Y^\alpha)$. Also we define

$$\mathcal{K}_i \equiv d\mathcal{A}_i + \mathcal{A}_i \wedge \mathcal{A}_i\,, \tag{4.8}$$

$$k_{ij} \equiv da_{ij} + a_{ij} \wedge a_{ij} + T_{\mathcal{A}_i}(a_{ij})\,. \tag{4.9}$$

A twisted 1-gerbe is a set $(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i, B_i, d_{ij}, H_i)$ such that, $\varphi_{ij}|_{\mathrm{U}(1)} = id$, $T_{\mathcal{A}_i}|_{\mathrm{U}(1)} = 0$,

$$\varphi_{ij}\varphi_{jk} = Ad_{f_{ijk}}\varphi_{ik}\,, \tag{4.10}$$

$$\mathcal{A}_i + ad_{a_{ij}} = \varphi_{ij}\mathcal{A}_j\varphi_{ij}^{-1} + \varphi_{ij}d\varphi_{ij}^{-1}\,, \tag{4.11}$$

$$d_{ij} + \varphi_{ij}(d_{jk}) = f_{ijk}d_{ik}f_{ijk}^{-1} + T_{\mathcal{K}_i + ad_{B_i}}(f_{ijk})\,, \tag{4.12}$$

$$\varphi_{ij}(H_j) = H_i + d\,d_{ij} + [a_{ij}, d_{ij}] + T_{\mathcal{K}_i + ad_{B_i}}(a_{ij}) - T_{\mathcal{A}_i}(d_{ij})\,, \tag{4.13}$$

and such that $\mathbf{D}_H(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i, B_i, d_{ij}) \equiv (\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i)$ has U(1)- and Lie(U(1))-valued elements, where

$$\lambda_{ijkl} \equiv f_{ikl}^{-1}f_{ijk}^{-1}\varphi_{ij}(f_{jkl})f_{ijl}\,, \tag{4.14}$$

$$\alpha_{ijk} \equiv a_{ij} + \varphi_{ij}(a_{jk}) - f_{ijk}a_{ik}f_{ijk}^{-1} - f_{ijk}df_{ijk}^{-1} - T_{\mathcal{A}_i}(f_{ijk})\,, \tag{4.15}$$

$$\beta_{ij} \equiv \varphi_{ij}(B_j) - B_i - d_{ij} + k_{ij}\,, \tag{4.16}$$

$$\gamma_i \equiv H_i - dB_i + T_{\mathcal{A}_i}(B_i)\,, \tag{4.17}$$

and we have used the same notation φ_{ij} for the induced map $\varphi_{ij} : O_{ij} \to Aut(\mathrm{Lie}(G))$.[3]

[3] In the special case $G = \mathrm{U}(1)$ the $\mathbf{D}_H$ operation is equivalent to the usual Deligne coboundary operator, provided we change $f \to f^{-1}, B \to -B$, $(d \to -d,\ H \to -H)$ and set $H = 0$.

If there is zero on the l.h.s. of equations (4.15)–(4.17) and 1 on the l.h.s. of eq. (4.14), equations (4.10)–(4.17) define a nonabelian gerbe[4]. A little algebra, see the appendix, shows that in the less trivial situation, when we assume that λ_{ijkl} is U(1)-valued and α_{ijk}, β_{ij} and γ_i are Lie(U(1))-valued, the above equations guarantee that $(\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i)$ is a honest 2-gerbe; hence the name twisted 1-gerbe for the set $(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i, B_i, d_{ij}, H_i)$.

The 2-gerbe may not satisfy the antisymmetry property in its indices. This however can always be achieved by a gauge transformation with a trivial Deligne class.

We say that the nonabelian gerbe $(f_{ijk}, \varphi_{ij}, a_{ij}, d_{ij}, A_i, B_i, H_i)$ is twisted by the 2-gerbe $(\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i)$. We can also say (compare to the one degree lower situation) that we have a 2-gerbe module, or that we have a lifting 2-gerbe. The name "lifting 2-gerbe" comes from the following observation: under the projection π, that enters the group extension $\mathrm{U}(1) \to G \xrightarrow{\pi} G/\mathrm{U}(1)$, the twisting 2-gerbe disappears and we are left with an ordinary $G/\mathrm{U}(1)$-nonabelian gerbe (for example the map φ_{ij} is now given by $\pi(\varphi(\hat{g}))$ and is independent from the lifting $\hat{g}$ of the element $g \in G/\mathrm{U}(1)$). The twisting 2-gerbe is the obstruction to lift the nonabelian $G/\mathrm{U}(1)$-gerbe to a G-gerbe.

4.1 Twisted $\tilde{\Omega}E_8$ gerbes

Consider the exact sequence of groups,

$$1 \to \Omega E_8 \to PE_8 \xrightarrow{\pi} E_8 \to 1\,, \tag{4.18}$$

where the loop group ΩE_8 is the space of loops based at the identity 1_{E_8}, and the based path group PE_8 is the space of paths starting at the identity 1_{E_8}.[5] Expression (4.18) states that ΩE_8 is a normal subgroup of PE_8, the quotient being E_8. Consider now the problem of lifting an E_8 bundle to a PE_8 bundle. Since every path can be homotopically deformed to the identity path, we have that PE_8 is contractible, and therefore every PE_8 bundle is the trivial bundle. This implies that only the trivial E_8 bundle can be lifted. Any nontrivial E_8 bundle cannot be lifted and we thus obtain a nontrivial ΩE_8 1-gerbe. If dim$M \leq 15$ (equivalence classes of) E_8 bundles are in 1-1 correspondence with elements $a \in H^4(M,\mathbb{Z})$ and we can say that a is the obstruction to lift the E_8 bundle, i.e. that a characterizes the gerbe. More explicitly the ΩE_8 gerbe has $Aut(\Omega E_8)$ valued maps φ_{ij} coming from the conjugation action by some $G_{ij} \in PE_8$ (these are the transition functions of the twisted PE_8 bundle associated with the ΩE_8 gerbe). Since ΩE_8 is normal in PE_8 also the actions $T_{\mathcal{A}_i}$ of the Lie($Aut(\Omega E_8)$) valued one forms $\mathcal{A}_i$ on $h \in \Omega E_8$ and $X \in$ Lie(ΩE_8) can be understood as $T_{\mathcal{A}_i}(h) = hA_ih^{-1} - A_i$ and $T_{\mathcal{A}_i}(X) = [X, A_i]$ with some Lie(PE_8)-valued forms A_i that are a lift of the connection on the E_8 bundle. (See [12] for more details on gerbes from an exact sequence $1 \to H \to G \xrightarrow{\pi} G/H \to 1$).

Finally consider the (universal) central extension of ΩE_8,

$$1 \to \mathrm{U}(1) \to \tilde{\Omega}E_8 \xrightarrow{\pi} \Omega E_8 \to 1\,, \tag{4.19}$$

[4]A general nonabelian 1-gerbe is defined by equations (4.10)-(4.17), where now the group G is an arbitrary group (not necessarily a central extension).

[5]More precisely we should use smooth loops and paths with sitting instant [32].

JHEP10(2004)068

and try to lift the ΩE_8 gerbe to an $\tilde{\Omega} E_8$ gerbe, this is in general not possible and the obstruction gives rise to a twisted $\tilde{\Omega} E_8$ gerbe, the twist being described by a 2-gerbe. Actually one has always an obstruction in lifting the ΩE_8 gerbe if at least $M \leq 14$, and therefore the lifting 2-gerbe thus obtained has characteristic class a (characterizing the initial E_8 bundle). The twisted $\tilde{\Omega} E_8$ gerbe has $Aut(\tilde{\Omega} E_8)$ valued maps φ_{ij}, obtained extending the previous φ_{ij} maps in such a way that they act trivially on the center U(1) of $\tilde{\Omega} E_8$, also the $T_{\mathcal{A}_i}$ map is similarly extended.

A similar statement holds for E_8 replaced by G_2, Spin_n where $n \geq 7$, F_4, E_6, E_7, when one correspondingly lowers the dimension of M.

5. M5-brane anomaly, 2-gerbes and twisted nonabelian 1-gerbes

In section 3 cancellation of global anomalies appearing in the open string worldsheet with strings ending on a stack of D-branes led to condition (3.13) for the D-brane configuration (charges). Here one could in principle follow a similar approach and study global anomalies of the path integral of open M2-branes ending on M5 branes. An alternative approach is to study anomalies of 11-dimensional supergravity in the presence of M5-branes. The relevant mechanism is the cancellation between anomalies of the M5 brane quantum effective action and anomaly inflow from the 11-dimensional bulk through a non invariance of the Chern-Simons plus Green-Schwarz topological term $\Phi(C) \sim \int \frac{1}{6} C \wedge G \wedge G - C I_8(g)$ where C is the 3-form potential of 11-dimensional supergravity, $G = dC$ and $I_8 \sim (\mathrm{Tr}\, R^2)^2 - \mathrm{Tr}\, R^4$, with R being the curvature. We are interested in the global aspects of this mechanism, where we cannot assume that C is globally defined and that G is topologically trivial. This problem has been studied in [13, 6, 7]; and in the more general case where the 11-dimensional space has boundaries in [8]. Let Y be the 11-dimensional spacetime: a spin manifold. Let also V be the six dimensional M5-brane worldvolume embedded in Y $\iota : V \hookrightarrow Y$, we assume it compact and oriented. It turns out [7] that if the field strength G is cohomologically trivial on V and V is the product space $V = S \times Q$, with S a circle with supersymmetric spin structure and Q a five manifold, then the M5-brane can wrap V iff Q is a Spin^c manifold. If this is not the case the M5-brane has a global anomaly: one detects it from the vanishing of the M5-brane partition function. The partition function is zero every time that there is a torsion element $\theta \in H^3_{tors}(Q, \mathbb{Z}) \subset H^3(Q, \mathbb{Z})$ different from zero. More in general, without assuming that $V = S \times Q$, we have a global anomaly if there exists an element $\theta \in H^4_{tors}(V, \mathbb{Z})$ different from zero. As suggested in [7] a way to cancel this anomaly is to turn on a background field G such that, essentially,

$$[G]|_V = \theta \,, \tag{5.1}$$

where $[G]|_V$ is the integral class associated with G restricted to V. This condition should be compared to (3.13) when the l.h.s. of (3.13) is zero: $[H]|_Q = W_3$. In the case $V = S \times Q$, dimensional reduction of the M5-brane on the circle S leads to a Type IIA D4-brane wrapping Q and satisfying $[H]|_Q = W_3$.

JHEP10(2004)068

In [8] condition (5.1) is sharpened. First a mathematically precise definition of C and of $\Phi(C)$ is given, it is in terms of connections on E_8 bundles. Associated with the field strength G on spacetime Y with metric g, there is an integral cohomology class $a \in H^4(Y,\mathbb{Z})$. This determines an E_8 bundle $P(a) \to Y$ (cf. section 2.3). The field C can then be described by a couple (A,c) where A is an E_8 connection on $P(a)$ and c is a globally defined $Lie(\mathrm{U}(1))$-valued 3-form on Y. We denote by $\check{C} = (A,c)$ this E_8 description of the C-field. In particular the holonomy of $\check{C}$ around a 3-cycle Σ is given as

$$\mathrm{hol}_\Sigma(\check{C}) = \exp\left[\left(\int_\Sigma CS(A) - \frac{1}{2}CS(\omega) + c\right)\right],$$

with properly normalized Chern-Simons terms corresponding to the gauge field A and the spin connection ω such that $\exp[(\int_\Sigma CS(A)]$ is well defined and $\exp[\frac{1}{2}(\int_\Sigma CS(\omega)]$ has a sign ambiguity. To be more precise these should be the holonomy of the E_8 Chern-Simons 2-gerbe and the proper square root of the holonomy of the Chern-Simons 2-gerbe associated with the metric.

Subsequently in [8] the electric charge associated with the C field is studied. From the C field equation of motion that are nonlinear, $d\star G = \frac{1}{2}G^2 - I_8$, we have that the C field and the background metric induce an electric charge that is given by the cohomology class

$$[\frac{1}{2}G^2 - I_8]_{DR} \in H^8(Y,\mathbb{R})\,. \tag{5.2}$$

However the electric charge is an integer cohomology class (because of Dirac quantization, due to the existence of fundamental electric M2-branes and magnetic M5-membranes). In [8] the integral lift of (5.2) is studied and denoted $\Theta_Y(\check{C})$ (and also $\Theta_Y(a)$).

In order to study the anomaly inflow, we consider a tubular neighbourhood of V in Y. Since this is diffeomorphic to the total space of the normal bundle $N \to V$, we identify these two spaces. Let $X = S_r(N)$ be the 10-dimensional sphere bundle of radius r; the fibres of $X \overset{\pi}{\to} V$ are then 4-speres. An 11-dimensional manifold Y_r with boundary X is then constructed by removing from Y the disc bundle $D_r(N)$ of radius r; $Y_r = Y - D_r(N)$ (we can also say that Y_r is the complement of the tubular neighbourhood $D_r(N)$). We call Y_r the bulk manifold. Then one has the bulk C field path integral $\Psi_{bulk}(\check{C}_X) \sim \int exp[G \wedge \star G]\Phi(\check{C}_{Y_r})$ where the integral is over all equivalence classes of $\check{C}_{Y_r}$ fields that on the boundary assume the fixed value $\check{C}_X$. The wavefunction $\Psi_{bulk}(\check{C}_X)$ is section of a line bundle $\mathcal{L}$ on the space of $\check{C}_X$ fields. This wavefunction appears together with the M5-brane partition function $\Psi_{M5}(\check{C}_V)$ that depends on the $\check{C}$ field on the M5-brane, or better, on an infinitesimally small ($r \to 0$) tubular neighbourhood of the M5-brane. Anomaly cancellation requires $\Psi_{bulk}\Psi_{M5}$ to be gauge invariant and therefore Ψ_{M5} has to be a section of the opposite bundle of $\mathcal{L}$ ($\check{C}$ fields on V and $\check{C}$ fields on X can be related according to the exact sequence (5.4)). Let's study the various cases.

I) We can have Ψ_{bulk} gauge invariant, and this is shown to imply $\Theta_Y(\check{C}_X) = 0$. This last condition is the decoupling condition, indeed if $\Theta_Y(\check{C}_X) \neq 0$ then charge conservation requires that M2-branes end on the M5-brane and the M5-brane is thus not decoupled from the bulk. If Ψ_{bulk} is gauge invariant also Ψ_{M5} needs to be, and this holds if $\theta = 0$.

II) More generally we can have $\theta \neq 0$ but then invariance of $\Psi_{bulk}\Psi_{M5}$ can be shown to imply

$$\pi_*(\Theta_X) = \theta\,, \tag{5.3}$$

where π_* is integration over the fibre. The map π_* enters the exact sequence

$$0 \to H^k(V,\mathbb{Z}) \stackrel{\pi^*}{\to} H^k(X,\mathbb{Z}) \stackrel{\pi_*}{\to} H^{k-4}(V,\mathbb{Z}) \to 0\,, \tag{5.4}$$

where π^* is just pull back associated with the bundle $X \stackrel{\pi}{\to} V$. The exactness of this sequence (obtained from the Gysin sequence) follows from X being oriented compact and spin, and V oriented and compact. Condition (5.3) is the precise version of condition (5.1).

We now compare this situation to that in 10 dimensional Type IIA theory, described at the end of section 3, and therefore we are led to consider the following more general case.

III) Here $\Psi_{bulk}\Psi_{M5}$ is not gauge invariant (therefore it is a section of a line bundle) but we can consider a new partition function Ψ'_{M5} that is obtained from a "stack" of M5-branes instead of just a single brane. This stack gives rise to a twisted gerbe $(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i, B_i, d_{ij}, H_i)$ on V so that in particular Ψ'_{M5} depends also from the nonabelian gauge fields B_i and H_i.[6] In order for $\Psi_{bulk}\Psi'_{M5}$ to be well defined, the twisted gerbe has to satisfy [cf. (3.12)],

$$[CS(\pi_*(\Theta_X))] - [\vartheta_{ijkl}, 0, 0, 0] = [\mathbf{D}_H(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i, B_i, d_{ij})] + [1, 0, 0, C_V]\,, \tag{5.5}$$

where, as constructed in subsection 2.3, $CS(\pi_*(\Theta_X))$ is the Chern-Simons 2-gerbe associated with $\pi_*(\Theta_X)$ and a choice of connection on the E_8 bundle with first Pontryagin class $\pi_*(\Theta_X)$ (all other 2-gerbes differ by a global 3-form, see (2.21)), while $[\vartheta_{ijkl}, 0, 0, 0]$ is the 2-gerbe class associated with the torsion class θ (i.e. $\beta(\vartheta) = \theta$, cf (2.24)), and $[1, 0, 0, C_V]$ is the trivial Deligne class associated with the global 3-form C_V.

In particular (5.5) implies

$$\pi_*(\Theta_X) - \theta = \xi_{\mathbf{D}_H(G_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i, B_i, d_{ij})}\,, \tag{5.6}$$

where on the r.h.s. we have the characteristic class of the lifting 2-gerbe.

The correspondence of this construction with that described in section 3, is strengthened by slightly generalizing the results of section 3. In fact there we always considered $[H]|_Q - W_3$ trivial in De Rham cohomology. This implied that the torsion class $[H]|_Q - W_3$ was interpreted as the characteristic class of a gerbe associated with a twisted $\mathrm{U}(n)$ bundle for some $n \in \mathbb{Z}$. However (at least mathematically) one can consider the more general case

[6] Of course we have the special case when a stack of M5-branes gives a nonabelian gerbe. Then H_i is the curvature of B_i. These two fields should not be confused, and have nothing to do with the NS B field and its curvature H.

where $[H]|_Q - W_3 \neq 0$ also in De Rham cohomology. Here too we have a twisted bundle, but with structure group $\mathrm{U}(\mathcal{H})$, the group of unitary operators on the complex, separable and infinite dimensional Hilbert space $\mathcal{H}$. This case corresponds to an infinite number of D-branes wrapping the cycle Q, and the relevant central extension is $\mathrm{U}(1) \to \mathrm{U}(\mathcal{H}) \to \mathrm{PU}(\mathcal{H})$. When $\dim Q \leq 13$ (which is always the case in superstring theory), we can replace, for homotopy purposes, $\mathrm{PU}(\mathcal{H})$ with ΩE_8 and $\mathrm{U}(\mathcal{H})$ with $\tilde{\Omega} E_8$, so that the group extension $\mathrm{U}(1) \to \mathrm{U}(\mathcal{H}) \to \mathrm{PU}(\mathcal{H})$ is replaced with $\mathrm{U}(1) \to \tilde{\Omega} E_8 \to \Omega E_8$. Now consider a stack of M5-branes wrapping a cycle $V = S \times Q$ and dimensionally reduce M-theory to Type IIA along the circle S. Then the M5-branes become D4-branes and the twisted ΩE_8 1-gerbe becomes a twisted ΩE_8 bundle.

Acknowledgments

We would like to thank Peter Bouwknegt, Dale Husemoller, Jouko Mickelsson and Stefan Theisen for useful discussions. P.A. would like to thank Masud Chaichian for the hospitality at University of Helsinki during completion of this work, and the financial support of the Academy of Finland under Projects No. 54023 and 104368.

A. Proof that a twisted 1-gerbe defines a 2-gerbe

The cocycle condition for λ_{ijkl} is straightforward. In order to show that α_{ijk} as defined in (4.15) satisfies the 2-gerbe condition

$$\alpha_{ijk} + \alpha_{ikl} - \alpha_{ijl} - \alpha_{jkl} = \lambda_{ijkl} d\lambda^{-1}_{ijkl}\,,$$

we rewrite the l.h.s. as $\alpha_{ijk} + Ad_{f_{ijk}}\alpha_{ikl} - Ad_{\varphi_{ij}(f_{jkl})}\alpha_{ijl} - \alpha_{jkl}$, we then use the definition of α_{ijk} and the following properties of the map $T_{\mathcal{A}}$,

$$T_{\mathcal{A}}(hk) = T_{\mathcal{A}}(h) + kT_{\mathcal{A}}(h)k^{-1}\,, \qquad \text{(cocycle property)} \tag{A.1}$$

$$\varphi_{ij}(T_{\mathcal{A}}(h)) = T_{\varphi_{ij}\mathcal{A}\varphi_{ij}^{-1}}(\varphi_{ij}(h))\,, \tag{A.2}$$

$$T_{-\varphi_{ij}d\varphi_{ij}^{-1}}(\varphi_{ij}(h)) = \varphi_{ij}(h)d\varphi_{ij}(h^{-1}) - \varphi_{ij}(hdh^{-1})\,, \tag{A.3}$$

where h, k are elements of G, and more in general functions from some open neighbourhood of M into G. Finally $T_{\mathcal{A}_i}(\varphi_{ij}(f_{jkl})f_{ijl}) = T_{\mathcal{A}_i}(f_{ijk}f_{ikl})$ since $T_{\mathcal{A}_i}(\lambda_{ijkl}) = 0$.

Similarly in order to show that

$$\beta_{ij} + \beta_{jk} + \beta_{ki} = d\alpha_{ijk}$$

we rewrite the l.h.s. as $\beta_{ij} + \varphi_{ij}(\beta_{jk}) + f_{ijk}\beta_{ki}f^{-1}_{ijk}$ and then use the following equality

$$k_{ij} + \varphi_{ij}(k_{jk}) = f_{ijk}k_{ik}f^{-1}_{ijk} + T_{\mathcal{K}_i}(f_{ijk}) + d\alpha_{ijk}\,, \tag{A.4}$$

that follows from (4.8)–(4.11), the algebra here is the same as for usual gerbes. We also have $\mathcal{K}_i + ad_{k_{ij}} = \varphi_{ij}\mathcal{K}_j\varphi^{-1}_{ij}$, and the Bianchi identity

$$dk_{ij} + [a_{ij}, k_{ij}] + T_{\mathcal{K}_i}(a_{ij}) - T_{\mathcal{A}_i}(k_{ij})\,. \tag{A.5}$$

Relation

$$\gamma_i - \gamma_j = d\beta_{ij}$$

that we rewrite as $\gamma_i - \varphi_{ij}(\gamma_j) = d\beta_{ij}$ follows from (A.5), (4.13) and $T_{-\varphi_{ij}d\varphi_{ij}^{-1}}(\varphi_{ij}(B_j)) = -d(\varphi_{ij}(B_j)) + \varphi_{ij}(dB_j)$.

References

[1] E. Witten, *D-branes and K-theory*, *J. High Energy Phys.* **12** (1998) 019 [hep-th/9810188].

[2] A. Kapustin, *D-branes in a topologically nontrivial B-field*, *Adv. Theor. Math. Phys.* **4** (2000) 127 [hep-th/9909089].

[3] P. Bouwknegt and V. Mathai, *D-branes, B-fields and twisted K-theory*, *J. High Energy Phys.* **03** (2000) 007 [hep-th/0002023].

[4] P. Bouwknegt, A.L. Carey, V. Mathai, M.K. Murray and D. Stevenson, *Twisted K-theory and K-theory of bundle gerbes*, *Commun. Math. Phys.* **228** (2002) 17 [hep-th/0106194].

[5] A.L. Carey, S. Johnson and M.K. Murray, *Holonomy on D-branes*, hep-th/0204199.

[6] E. Witten, *Five-brane effective action in M-theory*, *J. Geom. Phys.* **22** (1997) 103 [hep-th/9610234].

[7] E. Witten, *Duality relations among topological effects in string theory*, *J. High Energy Phys.* **05** (2000) 031 [hep-th/9912086].

[8] E. Diaconescu, G.W. Moore and D.S. Freed, *The M-theory 3-form and E_8 gauge theory*, hep-th/0312069.

[9] G.W. Moore, *Anomalies, Gauss laws and page charges in M-theory*, hep-th/0409158.

[10] M. J. Hopkins and I. M. Singer, *Quadratic functions in geometry, topology, and M-theory*, math.at/0211216.

[11] L. Breen, W. Messing *Differential geometry of gerbes*, math.AG/0106083.

[12] P. Aschieri, L. Cantini and B. Jurco, *Nonabelian bundle gerbes, their differential geometry and gauge theory*, hep-th/0312154.

[13] E. Witten, *On flux quantization in M-theory and the effective action*, *J. Geom. Phys.* **22** (1997) 1 [hep-th/9609122].

[14] D.E. Diaconescu, G.W. Moore and E. Witten, *A derivation of K-theory from M-theory*, hep-th/0005091.

[15] A. Adams and J. Evslin, *The loop group of E_8 and K-theory from 11D*, *J. High Energy Phys.* **02** (2003) 029 [hep-th/0203218];
J. Evslin, *Twisted K-theory from monodromies*, *J. High Energy Phys.* **05** (2003) 030 [hep-th/0302081]; *From E_8 to F via T*, *J. High Energy Phys.* **08** (2004) 021 [hep-th/0311235].

[16] V. Mathai and H. Sati, *Some relations between twisted K-theory and E_8 gauge theory*, *J. High Energy Phys.* **03** (2004) 016 [hep-th/0312033].

[17] P. Hořava and E. Witten, *Eleven-dimensional supergravity on a manifold with boundary*, *Nucl. Phys.* **B 475** (1996) 94 [hep-th/9603142].

[18] P.C. West, *E_{11} and M-theory*, *Class. and Quant. Grav.* **18** (2001) 4443 [hep-th/0104081].

[19] R. D'Auria and P. Fré, *Geometric supergravity in $D = 11$ and its hidden supergroup*, *Nucl. Phys.* **B 201** (1982) 101.

[20] I.A. Bandos, J.A. de Azcarraga, M. Picon and O. Varela, *On the formulation of $D = 11$ supergravity and the composite nature of its three-form field*, hep-th/0409100.

[21] N. Hitchin, *Lectures on special lagrangian submanifolds*, math.dg/9907034.

[22] J.L. Brylinski, *Loop spaces, characteristic classes and geometric quantization*, *Progress in mathematics*, **107**, Birkhäuser, Boston, 1993.

[23] J. Cheeger and J. Simons, *Differential characters and geometric invariants*, Stony Brook Preprint 1973, reprinted in *Lecture Notes in Mathematics* 1167, *Geometry and Topology Proc. 1983-84*, J. Alexander and J. Harer eds., Springer 1985.

[24] D.S. Freed and E. Witten, *Anomalies in string theory with D-branes*, hep-th/9907189.

[25] M. Bauer, G. Girardi, R. Stora and F. Thuillier, *A class of topological actions*, hep-th/0406221.

[26] P. Gajer, *Geometry of deligne cohomology*, *Invent. Math.* **127** no. 1 (1997) p. 155, alg-geom/96010259601025

[27] J. Mickelsson, *Current algebras and groups*, Plenum Press, New York, 1989.

[28] S. Johnson, *Constructions with bundle gerbes*, Ph.D. thesis, math.DG/0312175.

[29] E. Witten, *Topological tools in ten-dimensional physics*, *Int. J. Mod. Phys.* **A 1** (1986) 39.

[30] K. Gawędzki, *Topological actions in two-dimensional quantum field theories*, in *Nonperturbative quantum field theory*, G. 't Hooft et al. ed., Cargese 1987, 101-141, NATO *Adv. Sci. Inst. Ser. B: Phys.*, 185, Plenum Press, New York, 1988;
K. Gawędzki and N. Reis, *WZW branes and gerbes*, *Rev. Mod. Phys.* **14** (2002) 1281 [hep-th/0205233].

[31] A. Grothendieck, *Le groupe de Brauer, Séminaire Bourbaki*, vol. 9, exp 290, Soc. Math. France 1995.

[32] A. Caetano and R.F. Picken, *An axiomatic definition of holonomy*, *Int. J. Math.*, 5(6) 1994) 835, [IFM-14-93, see also KEK Library].

[33] M.K. Murray, *Bundle gerbes*, dg-ga/9407015.

JHEP10(2004)068